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1992 J. Phys. A: Math. Gen. 25 L633

(http://iopscience.iop.org/0305-4470/25/11/002)

J. Phys. A: Math. Gen. 25 (1992) L633–L637. Printed in the UK

LETTER TO THE EDITOR

Miura transforms for discrete Painlevé equations

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Received 19 February 1992

Abstract. A Miura transformation is presented relating the discrete Painlevé II equation to what plays the role of the modified d-P_{II} (equation (34) in the Painlevé–Gambier classification). The study of this transformation makes possible the derivation of the auto-Bäcklund transform of the discrete P_{II} and allows one to obtain particular solutions to the latter. Moreover the use of this transformation makes it possible to construct a new mapping, in the parameter space of P_{II}. The latter is a novel method for the construction of integrable discrete systems.

The Miura transform has been introduced for this paradigm of the nonlinear evolution equation, the KdV equation [1]. Starting from the equation for u: $u_t+6uu_x+u_{xxx}=0$ and transforming to a new dependent variable $u=v_x-v^2$ we obtain the modified KdV equation: $v_t-6v^2v_x+v_{xxx}=0$. The Miura transform (MT) is just a special kind of Bäcklund transform relating the solutions of two PDEs: once u is known one can obtain v through the solution of a Riccati equation. In a 'singularity analysis' [2] spirit MT transforms the double pole of the KdV to a simple pole for the mKdV. Miura transforms are known between quite a few pairs of nonlinear equations (although not all of them are identified by the label 'modified' in front of one of them).

Continuous Painlevé equations share several common properties with nonlinear PDEs [3]. This is by no means astonishing since all the Painlevé equations can be obtained as similarity reductions of some appropriate PDE [4]. These equations were discovered at the turn of the century when Painlevé and his school [5] classified all the equations of the form $w''=F(w',w,z)$ where F is polynomial in w', rational in w and analytic in z that are free from movable critical points, what we call in modern terminology the Painlevé property. Gambier provided the final classification of the Painlevé equations [5]. He obtained 50 different equations (within a Möbius transform of the dependent variable and a change of the independent one). He showed, further, that there are 24 equations of which 18 can be reduced to a quadrature or a linear ODE, while the remaining 6 are irreducible and define the Painlevé transcendents. The rest of the 50 equations can be deduced from the 24 'basic' ones, as shown by Gambier, through transformations that are nothing else but Bäcklund transformations. Moreover, the six basic Painlevé transcendents possess Bäcklund and auto-Bäcklund transformations [6], that make possible to relate the transcendents for various values of their parameters.

While the BT of Painlevé equations have been extensively studied, little is known about the possibility of existence of Miura transformations. Still one case exists where such a relation is known since the genesis of the Painlevé equations. It can be shown in a straightforward way that starting from the equation for P_{II}:

$$v''=2v^3+zv-(\alpha+1/2) \tag{1}$$

and introducing the transformation [7]:

$$\alpha u=v'+v^2+z/2 \tag{2a}$$

complemented by:

$$v=\frac{u'+1}{2u} \tag{2b}$$

one can obtain the equation:

$$u''=\frac{u'^2}{2u}+2\alpha u^2-zu-\frac{1}{2u}. \tag{3}$$

This last equation carries the number 34 in Gambier's classification. In this letter we shall refer to it as P34 or modified P_{II}. It is interesting to note at this point that the 'half'-Miura transform $(1, 2a)$, used by Gambier for the integration of P34, breaks down at $\alpha=0$. Still, the full Miura transform $(2a, b)$ can be used for integration even when $\alpha=0$. In that case $(2a)$, with LHS equal to zero, is just a Riccati, solved through $v=A'/A$ where A is any solution of the Airy equation $A''+(z/2)A=0$. Then $(2b)$ is a non-homogeneous linear equation for u whose solution is just $u=-A^2\int^z \mathrm{d}z/A^2$. In fact, the analysis of this equation is given, in a more general setting, within equation (27) of Gambier [5] while it is absent from Ince's book [7].

Discrete Painlevé equations have been discovered only recently. The first two, d-P_I [8, 9] and d-P_{II} [10], have appeared in physical applications, while the remaining three were obtained through the use of the new tool of singularity confinement [11] (that plays the role of the Painlevé method [2] for discrete systems). One can ask the natural question whether a Miura relation similar to (2) exists for the discrete P_{II}'s. We will show below that this is indeed the case.

The discrete P_{II} equation has the form:

$$y_{n+1}+y_{n-1}=\frac{y_n(an+b)+c}{y_n^2-1} \tag{4}$$

where a, b, c are constants. Its continuous limit, obtained by $y=\varepsilon v$, $an+b=-2-\varepsilon^2 z$, $c=\varepsilon^3(\alpha+1/2)$ as $\varepsilon\to 0$, is just (1). At this point one can naturally ask what is the form of the discrete P34 or modified d-P_{II}. In [12] we have explained that the form of the d-P's can be suggested by the Quispel [13] mappings:

$$f_3(x_n)x_{n-1}x_{n+1}-f_2(x_n)(x_{n-1}+x_{n+1})+f_1(x_n)=0 \tag{5}$$

where the f_i are in general quartic polynomials. Indeed, introducing the lattice parameter ε we write:

$$\begin{aligned} &x_n=x \\ &x_{n-1}x_{n+1}=x^2+\varepsilon^2(xx''-x'^2)+\mathcal{O}(\varepsilon^4) \\ &x_{n-1}+x_{n+1}=2x+\varepsilon^2x''+\mathcal{O}(\varepsilon^4) \end{aligned} \tag{6}$$

and thus if we aim at a specific Painlevé equation we must choose the f_i's so as to obtain the correct ratio between xx'' and x'^2. Since modified d-P_{II} belongs to the same type as P_{IV} its form should be [12]:

$$(x_{n-1}+x_n)(x_n+x_{n+1})=\frac{a_n x_n^2+m_n^2}{(x_n+z_n)}. \tag{7}$$

The precise form of the a_n, z_n, m_n can be obtained using the singularity confinement approach [11]. In fact, there are only two ways for the variable x of mapping (7) to diverge: either $x_{n+1}+x_n$ or x_n+z_n vanish. The consequences of the first singularity can be easily assessed. For $x_{n+1}+x_n$ to vanish we must have $a_n x_n^2+m_n^2=0$. Iterating the mapping once we obtain as a condition for $x_{n+1}+x_{n+2}$ to be finite the relation $a_{n+1}x_{n+1}^2+m_{n+1}^2=0$. This tells us that $a_{n+1}/a_n=m_{n+1}^2/m_n^2$ i.e. the ratio a/m^2 is constant. For the second type of singularity the analysis is implemented most simply if we start with $x_n+z_n=\lambda\varepsilon$ with λ normalized so as to lead to $x_{n+1}=1/\varepsilon$. Iterating, we find that x_{n+2} diverges as $1/\varepsilon$ and that the conditions for x_{n+3} to be finite read $a_{n+1}=a_{n+2}$, i.e. a_n is a constant (and thus m_n is also a constant), and $z_{n+1}+z_{n+2}=z_n+z_{n+3}$. The solution to this last equation is just $z_n=\alpha n+\beta+\gamma(-1)^n$. This gives the form of P34 in agreement with the results of [14].

The continuous limit of (7) can be found in a straightforward way. Putting $x=\varepsilon^2 u$, $z=z_0+\varepsilon^2 t$, $a=4z_0$, $m^2=\varepsilon^6\mu$ we find: $u''-u'^2/(2u)+2u^2/z_0+2tu/z_0-\mu/(2uz_0)=0$ i.e. precisely P34. Moreover, we can perform the coalescence limit of d-P34 by taking: $x_n=-z_n+\delta y_n$, $z_n=-a/2+\delta c/2-\delta^2\eta_n/a$, $m^2=-a(a-\delta c)^2/4$ leading to (at $\delta\to 0$): $x_{n-1}+x_n+x_{n+1}=c+\eta_n/x_n$, i.e. d-$P_I$.

We come now to the question of the existence of a Miura transformation relating d-P_{II} and d-P34. In the continuous case the solution of P34 was obtained through the solution of P_{II} via a Riccati equation. The discrete equivalent of the Riccati has been given by Hirota [15] (see also [16]): it is just a first-order homographic mapping. With this remark as a guide it is straightforward to find the MT of d-P_{II}. Starting from P34 (notice the choice of normalization):

$$(x_{n-1}+x_n)(x_n+x_{n+1})=\frac{-4x_n^2+m^2}{\lambda x_n+z_n} \tag{8}$$

we introduce the Miura transform:

$$\lambda x_n=(y_n-1)(y_{n+1}+1)-z_n \tag{9a}$$

and its complement:

$$y_n=\frac{m+x_{n-1}-x_n}{x_n+x_{n-1}} \tag{9b}$$

and find

$$y_{n+1}+y_{n-1}=\frac{y_n(z_{n-1}+z_n)+\lambda m+(z_n-z_{n-1})}{y_n^2-1}. \tag{10}$$

In fact, the Miura transformation is best represented as the mapping:

$$y_{n+1}=\frac{\lambda x_n+1+z_n-y_n}{y_n-1} \tag{11a}$$

$$x_{n+1}=\frac{m+(1-y_{n+1})x_n}{y_{n+1}+1} \tag{11b}$$

where, elimination of the x (resp y) variable results to d-P_{II} (resp d-P34). As we have seen in the introduction the integration of the $\alpha=0$ case is related to the Airy function solutions of P_{II}. The same applies here. In fact, taking $\lambda=0$, the first half of the Miura (9a) decouples. It is straightforward to show that if y_n is a solution of the first-order mapping

$$(y_n-1)(y_{n+1}+1)=z_n \tag{12}$$

then y_n also satisfies d-P_{II} (equation (10)) for $\lambda=0$. Equation (12) can be linearized following the techniques of [16]. Rewriting (12) as $y_{n+1}=(z_n+1-y_n)/y_n-1$ and putting $y_n=B_n/A_n$ we linearize (12) to:

$$\begin{aligned} A_{n+1}&=-B_n+A_n\\ B_{n+1}&=-(z_n+1)A_n+B_n \end{aligned} \tag{13}$$

or equivalently

$$A_{n+2}-2A_{n+1}+z_nA_n=0. \tag{14}$$

The latter is a discrete analogue of the Airy equation and once its solution is obtained we can construct y_n through $y_n=(A_n-A_{n+1})/A_n$. The second half of the MT (11b) can then be used to find x_n through the solution of a linear mapping in close parallel to the continuous case. Having dealt with the $\lambda=0$, in what follows we will assume $\lambda=1$.

As in the case of the continuous Painlevé equations the existence of a Miura relation is rich in consequences. Since P34 contains only m^2 choosing the plus or minus sign for m in (9b) would lead to solutions of P_{II} with $\pm m$ respectively. Defining $\psi_n=-y_n(-m)$ we find that, while y_n is a solution of P_{II} with the constant term (c in equation (4)) equal to $m+(z_n-z_{n-1})$, ψ_n corresponds to $m-(z_n-z_{n-1})$. (Recall: if $z_n=\alpha n+\beta$, then $z_n-z_{n-1}=\alpha$.) We have thus (eliminating x_n):

$$\psi_n=-y_n+\frac{2m}{y_n(y_{n+1}+y_{n-1})-y_{n+1}+y_{n-1}-(z_n+z_{n-1}+2)} \tag{15a}$$

$$y_n=-\psi_n+\frac{2m}{\psi_n(\psi_{n+1}+\psi_{n-1})+\psi_{n+1}-\psi_{n-1}-(z_n+z_{n-1}+2)}. \tag{15b}$$

From equation (15) we can construct easily the auto-Bäcklund transformation for d-P_{II}. Using d-P_{II} to eliminate y_{n-1} (ψ_{n-1}) from the RHS of (15) we obtain:

$$y_n+\psi_n=\frac{m(1-y_n)}{(y_{n+1}+1)(y_n-1)-z_n-m/2} \tag{16a}$$

$$y_n+\psi_n=\frac{m(1+\psi_n)}{(\psi_{n+1}-1)(\psi_n+1)-z_n+m/2}. \tag{16b}$$

System (16) is indeed the auto-Bäcklund [6] since eliminating either of the two variables gives d-P_{II} for the other. One interesting application of this transform is to generate particular solutions for d-P_{II}. Indeed, just as rational solutions of P_{II} are known for integer values of the parameter $(\alpha+1/2)$ in (1), rational solutions also exist for P_{II}. Starting from (10) with $m=z_{n-1}-z_n=-\alpha$, we remark that $y_n=0$ is a solution. Injecting this 'seed' solution into (16a) we obtain ψ_n, that is the solution for $m=-3\alpha$, as $\psi_n=2\alpha/(z_{n-1}+z_n+2)$. The iteration of the Bäcklund allows the construction of a rational solution for all the values of $m=\pm(2k+1)\alpha$. Similarly one can construct 'Airy'-type solutions for all values of $m=\pm 2k\alpha$ starting from the 'Airy' solution (14).

Next, we shift equation (16*b*) one step further in m and introduce $\chi_n \equiv y_n(m+2\alpha)$:

$$\chi_n + y_n = \frac{(m+2\alpha)(1+y_n)}{(y_{n+1}-1)(y_n+1)-z_n+m/2+\alpha}. \tag{16c}$$

Thus ψ_n, y_n and χ_n define a mapping Y in the variable m since they correspond to the values of $m-2\alpha$, m and $m+2\alpha$ respectively. Eliminating y_{n+1} between (16*a*) and (16*c*) we obtain this mapping explicitly:

$$Y_{m+2\alpha} = \frac{\begin{array}{l} Y_{m-2\alpha}(2Y_m^3 + Y_m^2\alpha - Y_m(2z-\alpha+2) - 2\alpha - m) + 2Y_m^4 \\ \qquad + Y_m^3(\alpha+m) - Y_m^2(2z-\alpha+2) - 2Y_m(\alpha+m) \end{array}}{\begin{array}{l} Y_{m-2\alpha}(-2Y_m^2 + Y_m(\alpha+m) + (2z-\alpha+2)) \\ \qquad -2Y_m^3 + Y_m^2\alpha + Y_m(2z-\alpha+2) + m \end{array}}. \tag{17}$$

Notice that at the autonomous limit ($\alpha=0$) this mapping is of the Quispel type. Its continuous limit can be found through: $Y = k+\varepsilon^2 u$, $m=-4k^3-kt\varepsilon^4$, $\alpha=-k\varepsilon^5/2$, with $k^2=(z+1)/3$ resulting to: $k(1-k^2)u''=6u^2+t$, i.e. P_I.

This is a most interesting result since it allows us to establish a close parallel with the situation for continuous systems. Not only does the Miura transform between d-P_{II} and d-P34 exist, but it can be used to obtain the auto-Bäcklund transform for d-P_{II}. This, in turn, can be used to generate particular solutions to d-P_{II}. Moreover, the auto-Bäcklund can be exploited in order to derive a new discrete equation, in the parameter space of d-P_{II}, just as the auto-Bäcklund of the continuous P_{II} leads to a discrete P_I equation. We have thus a new algorithm to be added to the arsenal of methods for producing integrable discrete equations [14], namely the use of discrete Bäcklund or Miura transforms. This makes the problem of finding the auto-Bäcklund relations of all the known discrete Painlevé equations even more interesting.

The authors are grateful to A Fokas, V Papageorgiou and F Nijhoff for stimulating discussions and exchange of correspondence.

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